Hamiltonian Analysis of $R + T^2$ Action

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We study the gravitational action which is a linear combination of the Hilbert-Palatini term and a term quadratic in torsion and possessing local Poincare invariance. Although this action yields the same equations of motion as General Relativity, the detailed Hamiltonian analysis without gauge fixing reveals some new points never shown in the Hilbert-Palatini formalism. These include that an additional term containing torsion appears in the spatial diffeomorphism constraint and that the primary second-class constraints have to be imposed in a manner different from that in the Hilbert-Palatini case. These results may provide valuable lessons for further study of Hamiltonian systems with torsion.

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I. INTRODUCTION

Among various attempts to look for a quantum gravity theory, gauge theories of gravity are very attractive since the idea of gauge invariance has already been successful in the foundation of other fundamental interactions. Local gauge invariance is a key concept in Yang-Mills theory. Together with Poincare symmetry, it lays the foundation of standard model in particle physics. Localization of Poincare symmetry leads to Poincare Gauge Theory(PGT) of gravity, which contains general relativity as a special case. In this theory, besides energy-momentum, the spin of matter fields is also introduced to take effect on gravitational dynamics. It turns out that from the perspective of PGT, in general, gravity is not only represented as curvature but also as torsion of space-time. (See [1] and references therein for a comprehensive account of torsion in gravity). A large number of actions which satisfy local Poincare symmetry have been analyzed by various researchers ([2] provides a comprehensive review and bibliography of the progress made in PGT).

Hamiltonian formalism is an inherent framework to study the dynamics of a physical theory. Although Hamiltonian analysis is performed for a large number of models in PGT, the results are at a formal level without explicit expressions of the additional required second-class constraints. From the point of view of canonical quantization, it is essential to have a well-defined consistent Hamiltonian theory at the classical level. Such an ingredient is missing if we want to incorporate torsion into candidate quantum gravity models. However, it has been shown [3, 4] that the connection dynamics of general relativity can be derived from an action which contains, apart from the standard Hilbert-Palatini term, a total derivative involving torsion known as the Nieh-Yan term.

In this paper, we are interested in an action which contains two terms, a standard Hilbert-Palatini term and a T^2 term which involves a product of two torsion tensors in a particular way. There are a couple of reasons for choosing this particular T^2 term. This term has not been analyzed in the literature of PGT. Also this term is actually the difference between the Holst term [5, 6] and Nieh-Yan term [7]. (For details see [3, 8].) Although it turns out that the only solutions of this action are torsion free, our analysis is still very interesting as we manage to explicitly determine all the second-class constraints unlike other attempts in PGT. There are several subtle and important points in our Hamiltonian analysis which may provide valuable lessons for further studies of Hamiltonian systems with torsion. In the

presence of torsion, an additional term containing torsion appears in the spatial diffeomorphism constraint. Moreover the primary second-class constraints appearing in the first-order formalism have to be imposed in a manner different from that in the Hilbert-Palatini theory without torsion.

The paper is organized as follows. We first perform the Lagrangian analysis based on action principle in Section II. Then in Section III, we perform the Hamiltonian analysis of the theory in a way which is very different from that in the Hilbert-Palatini theory as well as the one carried out in [3] for a related action. Our approach is more closely related to the approach taken in PGT [2]. This is the main section of the paper and our analysis indicates several new and important properties which may be useful for further analysis of any other action with torsion. Finally we end with a conclusion in Section IV.

We will restrict ourselves to 4 dimensions. The Greek letters $\mu, \nu \ldots$ refer to space-time indices while the uppercase Latin letters $I, J \ldots$ refer to the internal $SO(3,1)$ indices. Our spacetime metric signature is $(-+++)$. Later when we do the $3+1$ decomposition of spacetime, we will use the lowercase Latin letters from the beginning of the alphabet $a, b, \ldots$ to represent the spatial indices.

II. ACTION AND LAGRANGIAN ANALYSIS

The action of gravity which we consider in this paper reads

$$S = S_{HP} + \alpha S_T \tag{1}$$

where

$$S_{HP} = \int \mathrm{d}^4x \ eR = \int \mathrm{d}^4x e e_I^\mu e_J^\nu R_{\mu\nu}{}^{IJ}(\omega_\mu{}^{IJ}) \ ; \ S_T = \frac{1}{8}\int \mathrm{d}^4x \epsilon^{\mu\nu\rho\sigma}T^I{}_{\mu\nu}T_{I\rho\sigma}.$$

Here the coupling parameter α is a non-zero real number, e_I^μ is the tetrad, e denotes the absolute value of the determinant of the co-tetrad, $\omega_\mu{}^{IJ}$ is the spacetime spin-connection which is not torsion-free, and $\epsilon^{\mu\nu\rho\sigma}$ denotes the 4-dimensional Levi-Civita tensor density. Further

$$R_{\mu\nu}{}^{IJ} = \partial_{[\mu}\omega_{\nu]}{}^{IJ} + \omega_{[\mu}{}^{IK}\omega_{\nu]K}{}^J, \tag{2}$$

$$T^I{}_{\mu\nu} = \partial_{[\mu}e^I_{\nu]} + \omega_{[\mu}{}^I{}_{|J|}e^J_{\nu]} \tag{3}$$

are the definitions for curvature and torsion respectively [1]. This action is invariant under local Poincare transformations (see Appendix A). Moreover, it is easy to show that S_T can be rewritten as

$$S_T = \int \mathrm{d}^4x \left[-\frac{1}{8} e e_I^{[\mu} e_J^{\nu]} R_{\mu\nu\ KL}\ \epsilon^{IJKL} + \partial_\mu \left(\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} e_\nu^I D_{[\rho} e_{\sigma]I} \right) \right]$$
$$= \int -\mathcal{L}_H + \text{total derivative}$$

where $\mathcal{L}_H$ is known as the Holst term for generalized Palatini action of gravity. It is well known that, in the absence of matter, adding the Holst term to the Hilbert-Palatini action does not modify the Einstein's equations of motion [5, 6]. So, although the additional term in our action is not a total derivative, it is related by a total derivative, known as the Nieh-Yan term [7], to a term which does not modify the equations of motion. We now show that the action (1) also yields Einstein's equations as expected, and thus the coupling parameter α is free at classical level.

We will be working in the first-order formalism and hence both the co-tetrad e_μ^I and the spin connection $\omega_\mu^{\ IJ}$ are treated as independent fields. Also our covariant derivative D_μ acts in the following way:

$$D_\mu e_\nu^I := \partial_\mu e_\nu^I + \omega_\mu^{\ I}{}_J e_\nu^J.$$

Then the variation of the action (1) with respect to the basic variables give:

$$\delta(S_{HP}) = \int \mathrm{d}^4x \left[\left(e e_K^\alpha e_I^\mu e_J^\nu R_{\mu\nu}{}^{IJ} - 2 e e_I^\alpha e_K^\mu e_J^\nu R_{\mu\nu}{}^{IJ} \right) \delta e_\alpha^K \right.$$
$$\left. - \left(D_\mu \left[\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \epsilon_{IJKL} e_\rho^K e_\sigma^L \right] \right) \delta\omega_\nu^{\ IJ} \right], \tag{4}$$

$$\alpha\delta(S_T) = \int \mathrm{d}^4x \left[\frac{\alpha}{2} \left(D_\beta \left[\epsilon^{\alpha\beta\gamma\delta} D_{[\gamma} e_{\delta]K} \right] \right) \delta e_\alpha^K + \left(\frac{\alpha}{2} \epsilon^{\mu\nu\rho\sigma} e_{\nu J} D_{[\rho} e_{\sigma]I} \right) \delta\omega_\mu^{\ IJ} \right] \tag{5}$$

where we have dropped the total derivatives. Thus it is easy to determine the equations of motion of the full action. The variation of the spin connection $\omega_\mu^{\ IJ}$ yields

$$\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} e_\nu^K D_{[\rho} e_{\sigma]}^L \left[\frac{\alpha}{2} (\eta_{JK}\eta_{IL} - \eta_{IK}\eta_{JL}) - \epsilon_{IJKL} \right] = 0 \tag{6}$$

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[1] Our conventions of symmetrization and antisymmetrization are $A^{(ab)} := A^{ab} + A^{ba}$ and $A^{[ab]} := A^{ab} - A^{ba}$ respectively

Direct calculation shows that this implies $\epsilon^{\mu\nu\rho\sigma}e_\nu^K D_{[\rho}e_{\sigma]}^L = 0$, and hence similar to that in [9], the only solutions are $D_{[\rho}e_{\sigma]}^L = 0$. [2] So, in the absence of matter the only solutions are the torsion-free solutions.

On the other hand, the variation of the co-tetrad e_α^K leads to

$$ee_K^\alpha e_I^\mu e_J^\nu R_{\mu\nu}{}^{IJ} - 2ee_I^\alpha e_K^\mu e_J^\nu R_{\mu\nu}{}^{IJ} + \frac{\alpha}{2}\left(D_\beta[\epsilon^{\alpha\beta\gamma\delta}D_{[\gamma}e_{\delta]K}]\right) = 0 \qquad (7)$$

We then get back the standard Einstein's equation for co-tetrad from (7) after solving for spin connection by Eq.(6). Hence, although the action with which we started contains the T^2 term, the equations of motion are as same as those of the Hilbert-Palatini action. In the next section we shall perform the Hamiltonian analysis of this action.

III. HAMILTONIAN ANALYSIS

In the Hamiltonian formulation of Hilbert-Palatini theory the basic variables are the $SO(3,1)$ spin connection $\omega_a{}^{IJ}$ and its conjugate momentum. It is well known that this formulation contains second-class constraints. Since our action contains the other term which explicitly depends on torsion, we expect that there will be the other pair of conjugate variables and the second-class constraints will be somehow different from the Hilbert-Palatini case.

To seek a complete Hamiltonian analysis, we perform the $3+1$ decomposition of our fields without breaking the internal $SO(3,1)$ symmetry and also without fixing any gauge. To identify our configuration and momentum variables for performing Hamiltonian analysis, we can rewrite S_{HP} and S_T as:

$$S_{HP} = \int d^4x\left[ee_{[I}^t e_{J]}^a\left(\partial_t\omega_a{}^{IJ}\right) + ee_{[I}^t e_{J]}^a\left(-\partial_a\omega_t{}^{IJ} + \omega_{[t}{}^{IK}\omega_{a]}{}^{KJ}\right) + \frac{1}{2}ee_{[I}^a e_{J]}^b R_{ab}{}^{IJ}\right],$$
$$(8)$$

$$\alpha S_T = \alpha\int d^4x\left[\epsilon^{abc}D_b e_c^I\left(\partial_t e_a^I\right) + \epsilon^{abc}D_b e_c^I\left(-\partial_a e_t^I + \omega_{[t}{}^{IJ}e_{a]J}\right)\right] \qquad (9)$$

So we can identify the momenta with respect to $\omega_a{}^{IJ}$ and e_a^I as

$$\Pi_{IJ}^a := ee_{[I}^t e_{J]}^a \quad ; \quad \Pi_I^a := \alpha\epsilon^{abc}D_b e_{cI} \qquad (10)$$

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[2] Note that, although the covariant derivative in [9] is torsion free and different from our covariant derivative, this result still holds in the case there is an antisymmetrization in the spacetime indices.

where ϵ^{abc} denotes the 3-dimensional Levi-Civita tensor density. We further parametrize the tetrad and the co-tetrad fields as [9]:

$$e_{tI} = NN_I + N^a V_{aI} \quad ; \quad e^{tI} = -\frac{N^I}{N},$$

$$e_{aI} = V_{aI} \quad ; \quad e^{aI} = V^{aI} + \frac{N^a N^I}{N}, \tag{11}$$

with $\quad N^I V_{aI} = 0 \quad ; \quad N^I N_I = -1 \tag{12}$

What we have done is that we have reparametrized the 16 degrees of freedom of $e_{\mu I}$ into 20 fields given by (11) subject to the 4 constraints (12). From these definitions, the following identities also hold:

$$V^{aI} V_{bI} = \delta^a_b \quad ; \quad V^{aI} N_I = 0 \quad ; \quad N_a := V_{aI} V^I_b N^b,$$

$$V^{aI} V_a{}^J = \eta^{IJ} + N^I N^J \tag{13}$$

In terms of these fields the metric takes the standard form

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^a N_a & N_a \\ N_a & V_{aI} V^I_b \end{pmatrix}$$

It is easy to see that

$$g := \det(g_{\mu\nu}) = -N^2 \det(V_{aI} V^I_b),$$

$$e := \det(e_{\mu I}) = N\sqrt{\det(V_{aI} V^I_b)} = N\sqrt{\det(q_{ab})} \equiv N\sqrt{q},$$

where $q_{ab} := V_{aI} V^I_b$ is the induced 3-metric on spacelike hypersurfaces Σ_t in the $3+1$ decomposition of spacetime.

Using the definitions given above we can also prove the following two identities

$$- ee^a_{[I} e^b_{J]} = \frac{N^2}{e} \Pi^{[a}_{IK} \Pi^{b]}_{JL} \eta^{KL} + N^{[a} \Pi^{b]}_{IJ}, \tag{14}$$

$$V^a_I = -\frac{1}{\sqrt{q}} \Pi^a{}_{IJ} N^J \tag{15}$$

Using these identities, we can rewrite the actions as

$$S_{HP} = \int \mathrm{d}^4 x \left[\Pi^a_{IJ} \partial_t \omega_a{}^{IJ} - \left(\frac{N^2}{2e} \Pi^{[a}_{IK} \Pi^{b]}_{JL} \eta^{KL} R_{ab}{}^{IJ} + \frac{1}{2} N^{[a} \Pi^{b]}_{IJ} R_{ab}{}^{IJ} - \omega_t{}^{IJ} D_a \Pi^a_{IJ} \right) \right], \tag{16}$$

$$\alpha S_T = \int \mathrm{d}^4 x \left[\Pi^a_I \partial_t V^I_a + \left(NN^I D_a \Pi^a_I + N^a V^I_a D_b \Pi^b_I + \frac{1}{2} \omega_t{}^{IJ} \Pi^a_{[I} V_{J]a} \right) \right] \tag{17}$$

where the total derivative terms have been neglected. It is going to be clear that the torsion degrees of freedom are now encoded in Π_I^a. In particular, if $\Pi_I^a = 0$ the entire S_T vanishes and we are left with the torsion-free solutions. Also the fact that V_a^I is now a basic variable causes significant changes in the subsequent constraint analysis as compared to the Hilbert-Palatini case.

Now we are in a position to rewrite the total action in the ADM form

$$S_{HP} + \alpha S_T = \int \mathrm{d}^4 x \left[\Pi_{IJ}^a \partial_t \omega_a^{IJ} + \Pi_I^a \partial_t V_a^I - \left(NH + N^a H_a + \omega_t^{IJ} \mathcal{G}_{tIJ} \right) \right]$$

$$\text{where} \qquad H = \frac{1}{\sqrt{q}} \Pi_{IK}^a \Pi_{JL}^b \eta^{KL} R_{ab}^{IJ} - N^I D_a \Pi_I^a, \tag{18}$$

$$H_a = \Pi_{IJ}^b R_{ab}^{IJ} - V_a^I D_b \Pi_I^b, \tag{19}$$

$$\mathcal{G}_{tIJ} = -D_a \Pi_{IJ}^a - \frac{1}{2} \Pi_{[I}^a V_{J]a} \tag{20}$$

Subsequently we will drop the subscript t from $\mathcal{G}_{tIJ}$ and denote it as $\mathcal{G}_{IJ}$. The two pairs of conjugate variables are $\left(\Pi^a_{IJ}, \omega_a^{IJ} \right)$ and $\left(\Pi_I^a, V_a^I \right)$. The fundamental Poisson brackets are given by

$$\left\{ \omega_a^{IJ}(x), \Pi_{KL}^b(y) \right\} = \frac{1}{2} \delta_a^b \delta_K^{[I} \delta_L^{J]} \delta^3(x - y),$$
$$\left\{ V_a^I(x), \Pi_J^b(y) \right\} = \delta_a^b \delta_J^I \delta^3(x - y) \tag{21}$$

Now let us count the degrees of freedom. The pair $\left(\Pi^a_{IJ}, \omega_a^{IJ} \right)$ have 36 degrees of freedom while the pair $\left(\Pi_I^a, V_a^I \right)$ have 24. The total number of degrees of freedom are 60. In the Lagrangian analysis we have seen that the only solutions are the standard torsion-free solutions. Hence, if the Lagrangian and Hamiltonian formulations are to be equivalent, the constraints present in the theory should remove 56 degrees of freedom leaving only 4 per point. At this stage we have the following constraints:

- Since there is no momentum corresponding to ω_t^{IJ}, we have to impose 6 primary constraints $\Pi_{IJ}^t \approx 0$. This leads to 6 secondary constraints $\mathcal{G}_{IJ} \approx 0$.

- Also there is no momentum corresponding to e_t^I. We have to impose 4 constraints $\Pi_I^t \approx 0$ which lead to 3 secondary constraints $H_a \approx 0$ and 1 more secondary constraint $H \approx 0$.

- From Eqs (10), we can get two other sets of primary constraints

$$C_I^a := \Pi_I^a - \alpha \epsilon^{abc} D_b V_{cI} \approx 0, \tag{22}$$

$$\Phi^a_{IJ} := \Pi^a_{IJ} - \frac{1}{2}\epsilon^{abc}\epsilon_{IJKL}V^K_b V^L_c \approx 0 \tag{23}$$

From (22) we get 12 constraints while (23) gives 18 because of the antisymmetry in IJ.

The above constraints cannot all be first class. Also note that the constraints (22) and (23) are different from the second-class constraints considered in the Hamiltonian analysis of Hilbert-Palatini action [6, 9].

Before calculating the constraint algebra we note another interesting feature in our theory. While the Gauss constraint $\mathcal{G}_{IJ}$ generates the $SO(3,1)$ transformations, the constraint which actually generates the spatial diffeomorphisms is a combination given by

$$\tilde{H}_a := H_a + \omega_a{}^{IJ}\mathcal{G}_{IJ} + \frac{1}{\alpha}\epsilon_{abc}C^b_I\Pi^c_I \tag{24}$$

This can be easily demonstrated as:

$$\delta^{\tilde{H}_a}\omega_c{}^{IJ} := \left\{\omega_c{}^{IJ}, \tilde{H}_a(N^a)\right\} = N^a\partial_a\omega_c{}^{IJ} + \omega_a{}^{IJ}\partial_c N^a = \mathcal{L}_{N^a}\omega_c{}^{IJ},$$

$$\delta^{\tilde{H}_a}\Pi^c_{IJ} := \left\{\Pi^c_{IJ}, \tilde{H}_a(N^a)\right\} = N^a\partial_a\Pi^c_{IJ} - \Pi^a_{IJ}\partial_a N^c + \Pi^c_{IJ}\partial_a N^a = \mathcal{L}_{N^a}\Pi^c_{IJ},$$

$$\delta^{\tilde{H}_a}V^I_c := \left\{V^I_c, \tilde{H}_a(N^a)\right\} = N^a\partial_a V^I_c + V^I_a\partial_c N^a = \mathcal{L}_{N^a}V^I_c,$$

$$\delta^{\tilde{H}_a}\Pi^c_I := \left\{\Pi^c_I, \tilde{H}_a(N^a)\right\} = N^a\partial_a\Pi^c_I - \Pi^a_I\partial_a N^c + \Pi^c_I\partial_a N^a = \mathcal{L}_{N^a}\Pi^c_I \tag{25}$$

The geometrical meaning of $\tilde{H}_a$ makes it easy to calculate its Poisson brackets with other constraints. Including all the primary constraints found so far, we can write the total Hamiltonian as

$$H_T := NH + N^a\tilde{H}_a + \Lambda^{IJ}\mathcal{G}_{IJ} + \gamma^I_a C^a_I + \lambda^{IJ}_a\Phi^a_{IJ} \tag{26}$$

where the expressions of the constraints are given by equations (18),(24),(20),(22) and (23) respectively.

We now calculate the algebra generated by the constraints. It turns out that the terms which are not weakly zero are:

$$\left\{H(N), \Phi^a_{IJ}(\lambda^{IJ}_a)\right\} = -NN_I\Pi^a_J\left(\lambda^{IJ}_a + \frac{1}{\alpha}\epsilon_{IJKL}\lambda^{KL}_a\right), \tag{27}$$

$$\left\{C^a_I(\gamma^I_a), \Phi^b_{JK}(\lambda^{JK}_b)\right\} = \epsilon^{abc}\gamma^I_b V^J_c\left(\alpha\lambda^{IJ}_a + \epsilon_{IJKL}\lambda^{KL}_a\right) \tag{28}$$

At this point we can introduce further secondary constraints or try to solve for some of the Lagrangian multipliers. Before that, let us again consider the degrees of freedom. Clearly

$\mathcal{G}_{IJ}$ and $\tilde{H}_a$ are first class. Since the Hamiltonian formulation should be equivalent to the Lagrangian one for consistency, we would like H to also be first class although it is not at this moment. All together we would have $(6 + 3 + 1) = 10$ first-class constraints removing 20 degrees of freedom. The constraints C_I^a and Φ_{IJ}^a are second class removing $12 + 18 = 30$ degrees of freedom. To obtain the torsion-free case, we need to find 6 more constraints hidden in the Eqs (27) and (28). Note that for a consistent Hamiltonian system we require

$$\dot{\Phi}_{IJ}^a(\sigma_a^{IJ}) := \left\{ \Phi_{IJ}^a(\sigma_a^{IJ}), H_T \right\} \approx 0, \tag{29}$$

$$\dot{C}_I^a(\eta_a^I) := \left\{ C_I^a(\eta_a^I), H_T \right\} \approx 0 \tag{30}$$

for arbitrary smearing functions σ_a^{IJ} and η_a^I. Explicitly evaluating (29) using Eqs (27) and (28) we get

$$\dot{\Phi}_{IJ}^a \approx 0 \quad \Rightarrow -\epsilon^{abc}\gamma_{b[I}V_{J]c} + \frac{1}{\alpha}NN_{[I}\Pi_{J]}^a \approx 0 \tag{31}$$

$$\Rightarrow \left(\gamma_b^I V_c^J - \gamma_c^I V_b^J - \gamma_b^J V_c^I + \gamma_c^J V_b^I\right) - \epsilon_{abc}\frac{N}{\alpha}\left(N^I\Pi^{aJ} - N^J\Pi^{aI}\right) \approx 0 \tag{32}$$

Multiplying (32) with V_J^b and using the properties (13) we get

$$2\gamma_c^I + V_J^b\gamma_b^J V_c^I - V_J^b\gamma_c^J V_b^I + \epsilon_{abc}\frac{N}{\alpha}N^I\Pi^{aJ}V_J^b \approx 0 \tag{33}$$

By multiplying this equation with N_I, V_I^c and V_d^I respectively and using the relations (12) and (13) we obtain the following relations

$$\gamma_c^I N_I = \frac{N}{2\alpha}\epsilon_{abc}\Pi_J^a V_J^b, \tag{34}$$

$$\gamma_c^I V_I^c = 0, \tag{35}$$

$$\gamma_c^I V_d^I = 0 \tag{36}$$

where we have used Eq.(35) to obtain (36). Finally from the equations (34) and (36) we get a solution for the Lagrangian multiplier γ_c^I as

$$\gamma_c^I = -\frac{N}{2\alpha}\epsilon_{abc}N^I\Pi_J^a V_J^b \tag{37}$$

So, we have obtained 12 components of γ_a^I from the 18 equations in (31). Consequently there are 6 constraints remaining. By inserting the solutions (37) back into (31) we get the following constraint:

$$\Pi_J^a + \Pi_K^a N^K N^J + \Pi_K^b V^{aK}V_b^J \approx 0$$

It can be shown that this constraint is actually equivalent to

$$\Pi_I^a V_I^b + \Pi_I^b V_I^a \approx 0 \tag{38}$$

Rewriting this in terms of basic variables we get the desired 6 secondary constraints

$$\chi^{ab} := -\frac{1}{\sqrt{q}}\left(\Pi_I^{(a}\Pi^{b)IJ}N_J\right) \approx 0 \tag{39}$$

In the above analysis we have used the relations (12) and (13).

Note that something similar occurs in various analysis performed in PGT [10]. However, we have managed to explicitly extract all the second-class constraints. Of course, our constraint analysis is not over because we have to ensure that the enlarged constraint algebra is now closed. We cannot add further secondary constraints to close the algebra but have to consistently solve for the Lagrangian multipliers. It turns out that χ^{ab} have non-zero Poisson brackets with all constraints except for itself, Gauss and spatial diffeomorphism constraint. They are calculated as

$$\left\{\chi^{ab}(\sigma_{ab}), H(N)\right\} = \frac{\sigma_{ac}}{\sqrt{q}}\Pi_{[I}^a N_{J]}\left[-D_b\left(\frac{N}{\sqrt{q}}\Pi_{IL}^b\Pi_{JL}^c\right) + D_b\left(\frac{N}{\sqrt{q}}\Pi_{JL}^b\Pi_{IL}^c\right) - \frac{1}{2}NN_{[I}\Pi_{J]}^c\right]$$
$$+ \frac{2\sigma_{ab}}{\sqrt{q}}\Pi_K^a N_I V_J^c\Pi_{KJ}^b D_c(NN^I) - \frac{2\sigma_{cb}}{q}NN^J\Pi_{IJ}^b\Pi_{IK}^c D_a\Pi_K^a, \tag{40}$$

$$\left\{\chi^{ab}(\sigma_{ab}), C_I^c(\gamma_c^I)\right\} = \frac{\alpha\sigma_{ac}}{2\sqrt{q}}\Pi_{[I}^a N_{J]}\epsilon^{cdb}\gamma_d^{[I}V_b^{J]} + \frac{2\sigma_{ab}}{\sqrt{q}}\Pi_K^a N_I V_J^c\Pi_{KJ}^b\gamma_c^I - \frac{2\alpha\sigma_{cb}}{\sqrt{q}}N_J\Pi_{IJ}^b\epsilon^{cad}D_a\gamma_d^I,$$
$$\tag{41}$$

$$\left\{\chi^{ab}(\sigma_{ab}), \Phi_{IJ}^c(\lambda_c^{IJ})\right\} = \frac{2\sigma_{cb}}{\sqrt{q}}N_J\Pi_{IJ}^b\epsilon^{cad}\epsilon_{KLIM}V_d^M\lambda_a^{KL} \tag{42}$$

These along with (27) and (28) are the non-zero terms in the constraint algebra.

At this point we show a result which will be used a number of times subsequently to show the closure of the constraint algebra. Let ρ^{KL} be a function antisymmetric in the indices KL. We suppress the spatial indices as they are not important for this calculation. Suppose for any real number r we have

$$\rho^{MN}\epsilon_{MNIJ} + r\rho^{IJ} = 0 \tag{43}$$

It is easy to see that this implies

$$\rho^{IJ}\epsilon_{KLIJ} + \frac{1}{r}\rho^{MN}\epsilon_{MNIJ}\epsilon_{KLIJ} = 0$$

$$\text{or,}\quad -r\rho^{KL} - \frac{2}{r}\rho^{MN}\delta_M^{[K}\delta_N^{L]} = 0$$

$$\Rightarrow\quad \left(r + \frac{4}{r}\right)\rho^{IJ} = 0 \quad \Rightarrow\quad \rho^{IJ} = 0$$

Consequently we get the following result

$$\rho^{MN}\epsilon_{MNIJ} + r\rho^{IJ} = 0 \quad \Rightarrow \quad \rho^{IJ} = 0 \tag{44}$$

where we have used Eq.(43) and the properties of $SO(3,1)$ Levi-Civita symbols.

Also, it is possible to use the constraints C_I^a and Φ_{IJ}^a to derive a relation:

$$D_{IJ} := D_a\Pi_{IJ}^a - \frac{1}{\alpha}\epsilon_{IJKL}\Pi^{aK}V_a^L \approx 0 \tag{45}$$

From the Gauss constraint (20) and equation (45) and the result of (44) we get

$$\Pi_I^c V_{Jc} - \Pi_J^c V_{Ic} \approx 0 \tag{46}$$

Multiplying Eq.(46) with (a) N_J and then V_I^b and (b)V_I^b and then V_J^a, we get respectively

$$\Pi_I^b N^I \approx 0, \tag{47}$$

$$\Pi_I^a V_I^b - \Pi_I^b V_I^a \approx 0 \tag{48}$$

where we have again made use of the relations (12) and (13). From equations (38) and (48) we obtain $\Pi_I^a V_I^b \approx 0$. Multiplying this with V_b^J and using (47) and (13), we get $\Pi_I^a \approx 0$. So the constraints $\mathcal{G}_{IJ}, C_I^a, \Phi_{IJ}^a, \chi^{ab}$ together imply torsion free condition.

As a consequence, the constraint algebra is extremely simplified. It is easy to see that the Hamiltonian constraint now becomes first class because the brackets (27) and (40) are now weakly zero. Thus our previous wish can be fulfilled. The other three constraints remain second class. But the non-zero terms become much more simple as

$$\left\{C_I^a(\gamma_a^I), \Phi_{JK}^b(\lambda_b^{JK})\right\} = \epsilon^{abc}\gamma_b^I V_c^J\left(\alpha\lambda_a^{IJ} + \epsilon_{IJKL}\lambda_a^{KL}\right), \tag{49}$$

$$\left\{\chi^{ab}(\sigma_{ab}), C_I^c(\gamma_c^I)\right\} = -\frac{2\alpha\sigma_{cb}}{\sqrt{q}}N^J\Pi_{IJ}^b\epsilon^{cad}D_a\gamma_d^I, \tag{50}$$

$$\left\{\chi^{ab}(\sigma_{ab}), \Phi_{IJ}^c(\lambda_c^{IJ})\right\} = \frac{2\sigma_{cb}}{\sqrt{q}}N_J\Pi_{IJ}^b\epsilon^{cad}\epsilon_{KLIM}V_d^M\lambda_a^{KL} \tag{51}$$

It is not difficult to check that except for Eqs (49-51) the constraint algebra is weakly closed. We can easily solve the equations $\dot{C}_I^a \approx 0$, $\dot{\chi}^{ab} \approx 0$ and $\dot{\Phi}_{JK}^c \approx 0$ with arbitrary smearing functions and get the solutions of the Lagrange multipliers $\lambda_c^{JK} = 0$ and $\gamma_a^I \approx 0$. Thus we go back to Hamiltonian formulation of general relativity without torsion which has been well studied [9].

IV. CONCLUDING REMARKS

In general relativity, one usually works with torsion-free connections, and thus gravitational degrees of freedom are encoded only in the metric tensor. However, in general, from the perspective of local Poincare invariance, it is possible to introduce further gravitational degrees of freedom via an independent torsion tensor. Furthermore, there are models where torsion can be used to explain the current acceleration of our universe[11]. However, a well-defined dynamical theory requires a consistent Hamiltonian description. Moreover, to incorporate torsion into a theory of quantum gravity built from the canonical perspective, it is essential to have a well-defined Hamiltonian formalism. Our goal in this paper is to obtain a consistent Hamiltonian analysis via the Dirac procedure of a typical action containing torsion.

Our action contains a term quadratic in torsion apart from the Hilbert-Palatini term. In fact, this torsion term is just the difference between the Nieh-Yan term and the Holst term. Although the final result is the expected torsion-free case, we learnt several interesting lessons in our analysis, which may be important for further studies of torsion in the Hamiltonian framework. The key points of our analysis may be summarized as follows.

(i) We construct the spatial diffeomorphism constraint $\tilde{H}_a$ in equation (24). Unlike the torsion-free case, $\tilde{H}_a$ contains an additional term $\frac{1}{\alpha}\epsilon_{abc}C_I^b\Pi_I^c$ because of the torsion term in action (1). Without adding this term we cannot ensure that the diffeomorphism constraint generates Lie derivatives of the basic variables.

(ii) When we get six new second-class constraints (39) from the consistency condition of constraint Φ_{IJ}^a, we do two things at the same time. One is to solve for Lagrangian multiplier γ_c^I, while the other is to determine secondary constraint χ^{ab}. They are related to each other. This is certainly different from the analysis in Hilbert-Palatini theory [9] as well as the theory studied in [3]. We also manage to get an explicit expression of this new set of secondary second-class constraints which has not been done before.

(iii) We get $\Pi_I^a \approx 0$ from the constraints $\mathcal{G}_{IJ}, C_I^a, \Phi_{IJ}^a$ and χ^{ab} and prove that the constraint algebra is closed. The procedure of the proof is very interesting and may find application in other Hamiltonian systems.

(iv) We do not fix any gauge before performing the Hamiltonian analysis, because gauge fixing in systems with second-class constraints might give rise to inconsistencies and some equations of motion of original theory might be lost.

In this paper we demonstrate a procedure to obtain a consistent Hamiltonian analysis of the action with torsion. For future investigation, we may consider matter coupling in this action. For example, in the case there is Dirac field, torsion is non-vanishing. It will be interesting to analyze such an action following our procedure. We may also use the results of this paper to analyze actions like $R + R^2$ with non-zero torsion. Torsion is dynamical in such models even without matter coupling. Studying models with dynamical torsion by Hamiltonian analysis will give us a deeper understanding of the dynamical behaviour related to torsion. This may further shed light on how to incorporate torsion into theories of quantum gravity.

Appendix A: Proof of Local Poincare Invariance

In this section, we prove that the action (1) is invariant under local Poincare transformations. We shall prove this for the two terms, eR and $\epsilon^{\mu\nu\alpha\beta}T^I{}_{\mu\nu}T_{I\alpha\beta}$ separately.

At first we prove local translational symmetry of the two terms. The determinant of co-tetrad reads

$$e = \frac{1}{4!}\epsilon_{IJKL}\epsilon^{\mu\nu\alpha\beta}e_\mu^I e_\nu^J e_\alpha^K e_\beta^L \tag{A1}$$

Under infinitesimal local translational transformation, the co-tetrad, tetrad and spin connection transform respectively as [12]

$$\delta_0 e_\mu^K = -e_\lambda^K \partial_\mu \xi^\lambda - \xi^\lambda \partial_\lambda e_\mu^K,$$
$$\delta_0 e_K^\mu = e_K^\lambda \partial_\lambda \xi^\mu - \xi^\lambda \partial_\lambda e_K^\mu,$$

$$\delta_0 \omega_\mu^{IJ} = -\omega_\lambda^{IJ} \partial_\mu \xi^\lambda - \xi^\lambda \partial_\lambda \omega_\mu^{IJ}$$

where the gauge group parameters ξ^μ are functions of space-time points.

Up to first order we have [13]

$$e' = e - \xi^\lambda \partial_\lambda e - e \partial_\lambda \xi^\lambda, \tag{A2}$$

$$R'_{\mu\nu}{}^{KL} = R_{\mu\nu}{}^{KL} - \xi^\lambda \partial_\lambda R_{\mu\nu}{}^{KL} - R_{\mu\lambda}{}^{KL} \partial_\nu \xi^\lambda + R_{\nu\lambda}{}^{KL} \partial_\mu \xi^\lambda, \tag{A3}$$

$$T'^I_{\mu\nu} = T^I_{\mu\nu} - \xi^\lambda \partial_\lambda T^I_{\mu\nu} - T^I_{\lambda\nu} \partial_\mu \xi^\lambda + T^I_{\lambda\mu} \partial_\nu \xi^\lambda \tag{A4}$$

Under above transformations, $R = e_I^\mu e_J^\nu R_{\mu\nu}{}^{IJ}$ transforms to

$$R' = e_I'^\mu e_J'^\nu R'_{\mu\nu}{}^{IJ} = R - \xi^\lambda \partial_\lambda R. \tag{A5}$$

Therefore for the Hilbert-Palatini term we get

$$e' R' = eR - \partial_\lambda(\xi^\lambda eR). \tag{A6}$$

where the additional term is just a total divergence which can be ignored.

For the S_T term we have

$$\epsilon^{\mu\nu\alpha\beta} T'^I_{\mu\nu} T'_{I\alpha\beta} = \epsilon^{\mu\nu\alpha\beta} \left(T^I_{\mu\nu} - \xi^\lambda \partial_\lambda T^I_{\mu\nu} - T^I_{\lambda\nu} \partial_\mu \xi^\lambda + T^I_{\lambda\mu} \partial_\nu \xi^\lambda \right)$$

$$\left(T_{I\alpha\beta} - \xi^\sigma \partial_\sigma T_{I\alpha\beta} - T_{I\sigma\beta} \partial_\alpha \xi^\sigma + T_{I\sigma\alpha} \partial_\beta \xi^\sigma \right)$$

$$= \epsilon^{\mu\nu\alpha\beta} T^I_{\mu\nu} T_{I\alpha\beta} + \epsilon^{\mu\nu\alpha\beta} T^I_{\mu\nu} T_{I\alpha\beta} \partial_\lambda \xi^\lambda - 4\epsilon^{\mu\nu\alpha\beta} T^I_{\mu\nu} T_{I\lambda\beta} \partial_\alpha \xi^\lambda. \tag{A7}$$

Since

$$T_{I\lambda\beta} = \frac{1}{2} T_{I\rho\sigma} \delta_\lambda^{[\rho} \delta_\beta^{\sigma]} = -\frac{1}{4} T_{I\rho\sigma} \epsilon^{\rho\sigma\gamma\delta} \epsilon_{\lambda\beta\gamma\delta} \tag{A8}$$

we may rewrite the last term of (A7) as

$$4\epsilon^{\mu\nu\alpha\beta} T^I_{\mu\nu} T_{I\lambda\beta} \partial_\alpha \xi^\lambda = -\epsilon^{\mu\nu\alpha\beta} \epsilon_{\lambda\gamma\delta\beta} \epsilon^{\rho\sigma\gamma\delta} T^I_{\mu\nu} T_{I\rho\sigma} \partial_\alpha \xi^\lambda$$

$$= \delta_\lambda^{[\mu} \delta_\gamma^\nu \delta_\delta^{\alpha]} \epsilon^{\rho\sigma\gamma\delta} T^I_{\mu\nu} T_{I\rho\sigma} \partial_\alpha \xi^\lambda$$

$$= -4\epsilon^{\mu\nu\alpha\beta} T^I_{\mu\nu} T_{I\lambda\beta} \partial_\alpha \xi^\lambda + 2\epsilon^{\mu\nu\rho\sigma} T^I_{\mu\nu} T_{I\rho\sigma} \partial_\lambda \xi^\lambda$$

which implies that

$$\epsilon^{\mu\nu\alpha\beta} T^I_{\mu\nu} T_{I\alpha\beta} \partial_\lambda \xi^\lambda = 4\epsilon^{\mu\nu\alpha\beta} T^I_{\mu\nu} T_{I\lambda\beta} \partial_\alpha \xi^\lambda \tag{A9}$$

Putting this back into (A7) we see that the last two terms of (A7) vanish thereby giving the desired result. Hence, we can say $eR = e e_I^\mu e_J^\nu R_{\mu\nu}{}^{IJ}$ and $\epsilon^{\mu\nu\alpha\beta} T^I{}_{\mu\nu} T_{I\alpha\beta}$ are local translational invariant.

Next, we give proof on local Lorentz symmetry. Under infinitesimal local Lorentz transformations, the co-tetrad, tetrad and spin connection transform as

$$\delta_0 e_\mu^K = A^K{}_S e_\mu^S, \tag{A10}$$

$$\delta_0 e_K^\mu = A_K{}^S e_S^\mu, \tag{A11}$$

$$\delta_0 \omega_\mu^{IJ} = A^I{}_K \omega_\mu^{KJ} + A^J{}_K \omega_\mu^{IK} - \partial_\mu A^{IJ} \tag{A12}$$

where gauge group parameters A^{IJ} are also functions of space-time points, and $A^{IJ} = -A^{JI}$. It is obvious that e is invariant under rotation (A10), while the curvature transforms to [13]

$$R'_{\mu\nu}{}^{IJ} = R_{\mu\nu}{}^{IJ} + A^I{}_M R_{\mu\nu}{}^{MJ} + A^J{}_M R_{\mu\nu}{}^{IM} \tag{A13}$$

The term $eR = e e_I^\mu e_J^\nu R_{\mu\nu}{}^{IJ}$ has local Lorentz symmetry since

$$R' = e_I'^\mu e_J'^\nu R'_{\mu\nu}{}^{IJ} = R \tag{A14}$$

while the torsion transforms to [13]

$$T'^I{}_{\mu\nu} = T^I{}_{\mu\nu} + A^I{}_J T^J{}_{\mu\nu} \tag{A15}$$

Using Eq.(A15) we can obtain the invariance of the T^2 term under local Lorentz rotations as

$$\epsilon'^{\mu\nu\alpha\beta} T'^I{}_{\mu\nu} T'_{I\alpha\beta} = \epsilon^{\mu\nu\alpha\beta} (T^I{}_{\mu\nu} + A^I{}_J T^J{}_{\mu\nu})(T_{I\alpha\beta} + A_I{}^K T_{K\alpha\beta})$$
$$= \epsilon^{\mu\nu\alpha\beta} T^I{}_{\mu\nu} T_{I\alpha\beta}, \tag{A16}$$

In summary, the action (1) is invariant under local Poincare transformation, which gives a well-defined Poincare gauge theory of gravity.

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